ON THE ROLE OF SUPERCOMPACT AND EXTENDIBLE CARDINALS IN LOGIC

BY

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ABSTRACT

It is proved that the existence of supercompact cardinal is equivalent to a certain Skolem-Löwenheim Theorem for second order logic, whereas the existence of extendible cardinal is equivalent to a certain compactness theorem for that logic. It is also proved that a certain axiom schema related to model theory implies the existence of many extendible cardinals.

We assume that the reader is familiar with the notions of a supercompact and a β-supercompact cardinal and a normal ultrafilter over $P_k(\beta)$ (n.u.f.), defined in [4] (see also [2]), as well as with the usual notation of Set Theory.

THEOREM 1. *There exists a supercompact cardinal iff there is a μ_0 such that for all $R(\beta)$, $\beta \geqq \mu_0$ there is an $\alpha < \beta$ such that $\langle R(\alpha), \varepsilon \rangle$ can be elementarily embedded in $\langle R(\beta), \varepsilon \rangle$.*

The least such μ_0 is the first supercompact cardinal.

We begin the proof of the Theorem by a series of Lemmata.

LEMMA 1. *Let $\beta \geqq \gamma > \kappa$; let κ be α-supercompact for all $\alpha < \gamma$ and let γ be β-supercompact; then κ is β-supercompact.*

PROOF. We form the ultrapower $V^{P_\gamma(\beta)}/U$ where U is a normal ultrafilter over $P_\gamma(\beta)$. Let M_u be the transitive isomorph of $V^{P_\gamma(\beta)}/U$ and let * be the canonical elementary embedding of V into M_u. It is well known (see [4], and [2]) that any set which is hereditarily of cardinality $\leqq |P_\gamma(\beta)|$ is a member of M_u, as well as any subset of M_u of cardinality $\leqq |P_\gamma(\beta)|$. Hence, every subset of $P_\kappa(\beta)$ is in M_u. $\kappa^* = \kappa$ because $\kappa < \gamma$, and as is well known $\beta < \gamma^*$.

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$V \models \kappa$ is α-supercompact for all $\alpha < \gamma$. Hence, because $\kappa^* = \kappa$ $M_u \models \kappa$ is α-supercompact for all $\alpha < \gamma^*$. Since $\beta < \gamma^*$ we have $M_u \models \kappa$ is β-supercompact.

By definition of β-supercompactness we get $M_u \models$ there is a normal ultrafilter U' over $P_\kappa(\beta)$. We have already mentioned that $P(P_\kappa(\beta)) \in M_u$, hence U' is a normal ultrafilter over $P_\kappa(\beta)$ also in V.

LEMMA 2. *Let j be elementary embedding of $\langle R(\alpha), \varepsilon\rangle$ into $\langle R(\gamma), \varepsilon\rangle$ where $\alpha < \gamma$ and α and γ are limit ordinals. If j is not the identity on $R(\alpha)$ then, the first ordinal $\kappa < \alpha$ moved by j is δ-supercompact for all $\kappa \leqq \delta < \alpha$.*

PROOF. If j is not the identity, it must move some ordinal, (since if $j(\lambda) = \lambda$ for every $\lambda < \alpha$ then the rank of $j(x)$ equals the rank of x for every $x \in R(\alpha)$, and we prove by induction on the rank of x that $j(x) = x$ for every $x \in R(\alpha)$) let κ be the first ordinal moved by j. Let $\rho = \alpha \cap j(\kappa)$. We shall see that κ is δ-supercompact for all $\delta < \rho$.

For a cardinal $\delta < \rho$ define a normal ultrafilter U over $P_\kappa(\delta)$ by:

$$A \in U \leftrightarrow \{j(i) \mid i < \delta\} \in j(A).$$

($j(A)$ is always defined because $P(P_\kappa(\delta)) \in R(\alpha)$, α being a limit ordinal).

Note that $\delta < j(\kappa)$; hence, $\left| \{j(i) \mid i < \delta\} \right| = \delta < j(\kappa)$ which means $\{j(i) \mid i < \delta\} \in P_{j(\kappa)}(j(\delta))$. It can be easily verified that U is indeed a normal ultrafilter over $P_\kappa(\delta)$.

Let us start iterating the application of j to κ. Of course, we can do it as long as $j^{n-1}(\kappa)$ is still in $R(\alpha)$. We shall use an argument of Kunen [1] to prove that $j^n(\kappa)$ is undefined for some $n < \omega$, since $j^{n-1}(\kappa) \geqq \alpha$, or $\sup_{n<\omega} j^n(\kappa) = \alpha$.

Suppose $j^n(\kappa)$ is defined for all $n < \omega$. It means $j^n(\kappa) < \alpha$. Let $\mu = \sup_{n<\omega} j^n(\kappa)$. And assume $\mu < \alpha$.

We shall now prove $j(\mu) = \mu$. Denote $B = \{j^n(\kappa) \mid n < \omega\}$, then $j(B = \{j^{n+1}(\kappa) \mid n < \omega\}$. Hence, $j(\mu) = \sup j(B) = \sup B = \mu$.

We get now a contradiction by using the following theorem of Erdös: There is a function F from $P^\omega(\mu)$, which is the set of all subsets of μ of cardinality ω, such that for any subset A of μ, of cardinality μ $F[P^\omega(A)] = \mu$. $F \in R(\alpha)$ because $\mu < \alpha$ and α is a limit ordinal.

Let $A = \{j(\eta) \mid \eta < \mu\}$. Clearly, $A \subseteq \mu$. $R(\alpha) \models F$ is a function from $P^\omega(\mu)$ to μ such that for any subset B of μ of cardinality μ, $F[P^\omega(B)] = \mu$. Since j is an elementary embedding and $j(\mu) = \mu$, $R(\gamma) \models j(F)$ is a function from $P^\omega(\mu)$ to μ such that for any subset B of μ of cardinality μ $F[P^\omega(B)] = \mu$. Hence,

$$j(F)[\boldsymbol{P}^{\omega}(A)] = \mu$$

which implies

$$\exists t[t \in \boldsymbol{P}^{\omega}(A) \text{ and } j(F)(t) = \kappa].$$

Let t be a subset of $A = \{j(\eta) \mid \eta < \mu\}$ such that $j(F)(t) = \kappa$ and let $t' = \{\beta \mid j(\beta) \in t\}$.

It is evident that $t = j(t')$; so, $j(F)(j(t')) = \kappa$. That is, $j(F(t')) = \kappa$; but κ, being the first ordinal moved by j, is not in the image of j, which is a contradiction.

Let n_0 be the biggest natural number such that $j^{n_0}(\kappa)$ is defined. We shall now show by induction on i that κ is δ-supercompact for all $\delta < j^i(\kappa) \cap \alpha$ $1 \leqq i \leqq n_0$. We have already proved it for $i = 1$.

Suppose we know already that κ is δ-supercompact for all $\delta < j^i(\kappa) \cap \alpha$ and $i + 1 \leqq n_0$ (which means $j^i(\kappa) < \alpha$). Let $x = j^i(\kappa)$. $R(\alpha) \models \kappa$ is δ-supercompact for all $\delta < x$, because being δ-supercompact relativizes to $R(\alpha)$.

Since j is an elementary embedding, we have

1) $R(\gamma) \models j(\kappa)$ is δ-supercompact for all $\delta < j(x)$. We deal in the case $i + 1 \leqq n_0$, so $1 < n_0$ and $j(\kappa) \cap \alpha = j(\kappa)$; thus, we know already that

2) $R(\gamma) \models \kappa$ is δ-supercompact for all $\delta < j(\kappa)$.

Combining (1) and (2) we get, by Lemma 1,

$$R(\gamma) \models \kappa \text{ is } \delta\text{-supercompact for all } \delta < j^{i+1}(\kappa).$$

In any case, whether $n_0 = 1$ or not, we get that

$$\kappa \text{ is } \delta\text{-supercompact for all } \delta < \alpha \cap j^{n_0}(\kappa).$$

But $j^{n_0}(\kappa) \geqq \alpha$. Otherwise, $j^{n_0}(\kappa)$ would be defined.

Therefore, κ is δ-supercompact for all $\delta < \alpha$ which was to be proved.

That finished the case where $j^n(\kappa)$ were undefined for some n. In the other case $\sup_{n<\omega} j^n(\kappa) = \alpha$ and by the same inductive proof we can show that κ is δ-supercompact for all $\delta < j^n(\kappa)$. Thus κ is supercompact for $\delta < \alpha$.

Note. In the original proof we assumed $\mathrm{cf}(\alpha) > \omega$ (where $\mathrm{cf}(\alpha)$ is the confinality index of α). The treatment of the case $\mathrm{cf}(\alpha) = \omega$ was given by the referee.

LEMMA 3. *If κ is a supercompact cardinal then for every $\beta \geqq \kappa$, there is an $\alpha < \kappa$ such that $R(\alpha)$ can be elementary embedded in $R(\beta)$.*

PROOF. Let $\gamma = |R(\beta)|$; let U be a normal ultrafilter over $P_\kappa(\gamma)$; let M_u and * be as in the proof of Lemma 1.

$R(\beta) \in M_u$ because it is hereditarily of cardinality $\leqq \gamma$. * restricted to $R(\beta)$ is a subset of M_u of cardinality γ so $* \mid R(\beta) \in M_u$. $* \mid R(\beta)$ is clearly an elementary embedding of $\langle R(\beta), \varepsilon \rangle$ to $\langle R^{M_u}(\beta^*), \varepsilon \rangle$. We get (using, of course, $R^{M_u}(\beta) = R(\beta)$)

$$M_u \models \text{there is elementary embedding of } \langle R(\beta), \varepsilon \rangle \text{ into } \langle R(\beta^*), \varepsilon \rangle.$$

Since $\beta \leqq \gamma < \kappa^*$ and * is elementary we have

$$V \models \text{there is } \alpha < \kappa \text{ such that } \langle R(\alpha), \varepsilon \rangle \text{ can be elementary embedded in } \langle R(\beta), \varepsilon \rangle.$$

PROOF OF THEOREM 1. a) If there exists a supercompact cardinal κ, by Lemma 3 we can take $\mu_0 = \kappa$.

b) Assume the existence of μ_0 satisfying the condition of the Theorem.

By the usual reflection principle (see [3]) we have $\beta \geqq \mu_0$, $\mathrm{cf}(\beta) = \aleph_1$ and if $\kappa < \beta$ and κ is δ-supercompact for all $\delta < \beta$ then κ is δ-supercompact for all δ, i.e., κ is supercompact.

By our assumption, there is an $\eta < \beta$ and a function j such that j is an elementary embedding of $\langle R(\eta), \varepsilon \rangle$ into $\langle R(\beta + 2), \varepsilon \rangle$. It is clear that $\eta = \alpha + 2$ for some ordinal α and $j(\alpha + 1) = \beta + 1$, because $\alpha + 1$, $\beta + 1$ are defined in $\langle R(\alpha + 2), \varepsilon \rangle$ and $\langle R(\beta + 2), \varepsilon \rangle$ respectively, by the same formula.

For the same reason $j(\alpha) = \beta$. $\mathrm{Cf}(\alpha) = \aleph_1$ because:

$R(\beta + 2) \models$ there is a function from $\aleph_1$ to β, unbounded in β. (It is clear that $j(\aleph_1) = \aleph_1$).

α is not the first ordinal moved by j, since if that were the case, then α would be measurable by means of the ultrafilter $U = \{A \mid A \subseteq \alpha, \alpha \in j(A)\}$, contradicting $\mathrm{cf}(\alpha) = \aleph_1$. Hence, j is not the identity on $R(\alpha)$, but it is an elementary embedding of $\langle R(\alpha), \varepsilon \rangle$ into $\langle R(\beta), \varepsilon \rangle$.

By Lemma 2 there is $\kappa < \alpha$ s.t. κ is δ-supercompact for all $\delta < \alpha$. Therefore,

$$R(\alpha) \models \kappa \text{ is } \delta\text{-supercompact for all } \delta < \alpha.$$

We get

$$R(\beta) \models j(\kappa) \text{ is } \delta\text{-supercompact for all } \delta < j(\alpha) = \beta,$$

but by definition of β $j(\kappa)$ is a supercompact cardinal.

Now we prove that the first μ_0 satisfying the condition of the theorem is the first supercompact cardinal. Let κ be the first supercompact cardinal. By Lemma 3 $\mu_0 \leqq \kappa$. Suppose $\mu_0 < \kappa$.

$R(\kappa) \models$ for all $\beta \geqq \mu_0$ there is $\alpha < \beta$ such that $\langle R(\alpha), \varepsilon \rangle$ can be elementary embedded in $\langle R(\beta), \varepsilon \rangle$. $\langle R(\kappa), \varepsilon \rangle$ is a model of set theory, so we can repeat our argument and get $R(\kappa) \models$. There is a supercompact cardinal κ_0, but $\kappa_0 < \kappa$ and κ_0 is δ-supercompact for all $\delta < \kappa$. By Lemma 1 κ_0 is supercompact, contradicting the minimality of κ. Q.E.D.

Theorem 1 is now used in order to characterize supercompactness in terms of a Skolem-Löwenheim property for higher order logic.

THEOREM 2. *The first supercompact cardinal is the first* μ_0 *such that for every structure* $A = \langle M, R_1, \cdots, R_n \rangle$ $|M| \geqq \mu_0$ *and every* Π^1_1*-sentence* ϕ*, such that* $A \models \phi$*, there exists a substructure* $A' = \langle M', R_1 | M' \cdots, R_n | M' \rangle$ *of* A *with* $|M'| < |M|$ *and* $A' \models \phi$.

LEMMA 4. *If* κ *is supercompact,* $A = \langle M, R_1, \cdots, R_n \rangle$ *a structure with* $|M| \geqq \kappa$*, and* ϕ *any finite order statement which holds in* A*, then there exists a substructure* $A' = \langle M', R_1 | M', \cdots, R_n | M' \rangle$ *of* A *with* $|M'| < \kappa$ *and* $A' \models \phi$.

PROOF. The proof is much like the proof of Lemma 3.

Let ω be of order m. $|M| = \gamma$. Denote $\delta = |R(\gamma + m)|$. We can assume without loss of generality that $M = \gamma$. Pick some n.u.f. on $P_\kappa(\delta)$ and build M_u and $*$. $A = \langle \gamma, R_1, \cdots, R_n \rangle$ is hereditarily of cardinality $\leqq \gamma$. Therefore, $A \in M_u$. $*$ restricted to γ clearly maps A on a substructure of A^*.

$M_u \models$ "$A \models \phi$" because ϕ is of order n and the power set of A of order m is in M_u ($\delta = |R(\gamma + m)|$) $|A^*|^{M_u} = \gamma^* \geqq \kappa^* > \gamma$. Therefore, $M_u \models$. There is a substructure of A^* of cardinality less than κ^* such that ϕ holds in it. Hence, $V \models$. There is a substructure of A of cardinality $< \kappa$ such that ϕ holds in it.

PROOF OF THEOREM 2. We prove that any μ_0 satisfying the condition of the present Theorem satisfies also the condition of Theorem 1; which, by Theorem 1 and Lemma 4, establishes the present Theorem.

Let $\phi(x, y)$ be the set theoretical statement, asserting that x is an ordinal and y is $R(x)$. We can form $\phi(x, y)$ such that it relativizes to any $\langle R(\alpha), \varepsilon \rangle$.

A transitive structure $A = \langle M, \varepsilon \rangle$ is isomorphic to $R(\alpha)$ iff $A \models \forall x(\mathrm{On}(x) \rightarrow \exists y \phi(x, y))$ (denote this statement by Ψ; $\mathrm{On}(x)$ means that x is an ordinal) and the following Π^1_1 statement holds in A:

$$\forall x \forall X [\forall y [y \hat{\in} X \rightarrow y \in x] \rightarrow \exists z \forall t [t \hat{\in} X \leftrightarrow t \in z]] \quad (\chi)$$

(Note the difference between $\in$ which is the relation in the model and $\hat{\in}$ which is the second order logic $\in$.)

Let θ be a first order statement which is the conjunction of the pairing axiom, infinity axiom, and extentionality. There is a first order statement Σ such that if

$\langle M, \varepsilon\rangle \models \theta$ then $\langle M, \varepsilon, T\rangle \models \Sigma \leftrightarrow T$ is a truth definition for $\langle M, \varepsilon\rangle$ (where T is a unary relation).

Let $\beta \geqq \mu_0$, where μ_0 is defined by the condition of the Theorem. Let $A = \langle R(\beta), \varepsilon, T\rangle$ where T is a truth definition for $\langle R(\beta), \varepsilon\rangle$. Clearly,

$$A \models \Psi \wedge \chi \wedge \theta \wedge \Sigma$$

which is equivalent to some Π_1^1 statement Φ.

By assumption on μ_0, there is a substructure $A' = \langle M, \varepsilon \,|\, M, T \,|\, M\rangle$ $|M| < |R(\beta)|$ such that $A' \models \Phi$.

$\langle M, \varepsilon \,|\, M\rangle$ is an elementary submodel of $\langle R(\beta), \varepsilon\rangle$ because $T \,|\, M$ is still a truth definition for $\langle M, \varepsilon \,|\, M\rangle$. But $\langle M, \varepsilon \,|\, M\rangle \models \Psi \wedge \chi$ and $\langle M, \varepsilon \,|\, M\rangle$ is isomorphic to some transitive structure. Hence, that transitive structure is of the form $\langle R(\alpha), \varepsilon\rangle$ where $\alpha < \beta$. So $\langle R(\alpha), \varepsilon\rangle$ can be elementary embedded in $R(\beta)$. Q.E.D.

We shall now show that a certain axiom schema is a very strong axiom of infinity, namely, it implies the existence of many supercompact cardinals. The axiom schema is:

(V) If $\phi(x)$ defines a proper class of structures in the same language, then there exist two members of the class such that one can be elementary embedded in the second.

This axiom schema is usually called Vopenka's principle. It was considered independently also by Keisler.

DEFINITION. κ is called α-extendible if $\alpha > \kappa$ and there is a $\beta > \alpha$ and an elementary embedding j of $\langle R(\alpha), \varepsilon\rangle$ into $\langle R(\beta), \varepsilon\rangle$ such that $j(\kappa) > \kappa$ and κ is the first ordinal moved by j.

κ is extendible if it is α extendible for all $\alpha > \kappa$.*

It is clear that if κ is α extendible it is β extendible for all $\kappa < \beta \leqq \alpha$.

By Lemma 2 if κ is extendible it is supercompact.

LEMMA 5. *Let κ be a supercompact cardinal $\alpha < \beta < \kappa$ and let α be β extendible, then α is β extendible in $R(\kappa)$. (i.e. with R_k taken to be the universe).*

PROOF. By the definition of β extendibility there is γ and an elementary embedding j of $\langle R(\beta), \varepsilon\rangle$ into $\langle R(\gamma), \varepsilon\rangle$ such that $j(\alpha) > \alpha$ and α is the first ordinal

* The author does not know who defined the notion of extendible cardinal. He found this notion in the referee report and all the results concerning extendible cardinals were added in the revised version.

moved by j. If $\gamma < \kappa$, then our claim indeed holds. If $\gamma \geqq \kappa$ then by a proof completely analogous to Lemmata 3 and 4 we can prove the existence of $\gamma' < \kappa$ and j such that j is elementary embedding of $\langle R(\beta), \varepsilon\rangle$ into $\langle R(\gamma'), \varepsilon\rangle$ with $j(\alpha) > \alpha$ and α the first ordinal moved.

The proof is by taking a normal ultrafilter u on $P_\kappa(|R(\gamma)|)$ and taking the transitive isomorph of $V^{P_\kappa(|R(\gamma)|)}/u$, M_u and the canonical elementary embedding * of V into M_u. $R(\gamma) \in M_u$ and $j \in M_u$ but $\kappa^* > |R(\gamma)| \geqq \gamma$ and $\alpha^* = \alpha$ so in $M_u \models$ there is elementary embedding of $\langle R(\alpha^*), \varepsilon\rangle$ into $\langle R(\gamma), \varepsilon\rangle$ for some $\gamma < \kappa^*$ such that $j(\alpha^*) > \alpha^*$ and α^* the first ordinal moved.

By * being elementary embedding we get the result.

Let V' be the following axiom schema: (V') if the formula $\tau(x)$ defines a closed unbounded class of ordinals, then there is extendible cardinal in this class. (i.e. "the class of all extendible cardinals is "stationary").

THEOREM 3. *(V) implies (V') but it is not equivalent to it.*

Note. When this paper was submitted for publication (V') was like the present (V') except that it had "supercompact" in place of "extendible". The referee noted that the proof could be adapted to the present stronger version of V'. He also informed me that the fact that V' implies the existence of an extendible cardinal was noted by several people independently.

PROOF. Let $\tau(x)$ define a closed unbounded class of ordinals and F be the normal function enumerating this class. Let C be the class of all ordinals α which satisfy the following (1)–(4)

1) α is closed under F.

2) If $\beta < \alpha$ and there is $\gamma > \beta$ such that β is not γ extendible, then there is such $\gamma < \alpha$.

3) If $\beta, \delta < \alpha$ and β is δ extendible then there is $\delta' < \alpha$ and an elementary embedding j of $\langle R(\delta), \varepsilon\rangle$ into $\langle R(\delta'), \varepsilon\rangle$ such that $j(\beta) > \beta$ and β is the first ordinal moved by j.

4) $\mathrm{cf}(\alpha) = \aleph_1$.

C contains arbitrary large ordinals by the reflections principles [3].

Let us consider $C' = \{\langle R(\beta+2), \varepsilon, F \cap \beta\rangle \mid \beta \in C\}$ which is a proper class of structures in the same languages. By (V) there are $\alpha, \beta, \alpha < \beta$ and a function j such that j is an elementary embedding of $\langle R(\alpha+2), \varepsilon, F|\alpha\rangle$ into $\langle R(\beta+2), \varepsilon, F|\beta\rangle$. Let κ be the least ordinal moved by j; there is such since

$j(\alpha+1) = \beta+1$. As in the proof of Theorem 1, $\kappa < \alpha$ and κ is δ-extendible for all $\kappa \leqq \delta < \alpha$. By definition of C, κ is extendible. We shall also prove that κ is a fixed point of F. Suppose that this is not the case. Let μ be the last ordinal $< \kappa$ in the image of F, if there is such an ordinal and 0 otherwise. F is normal. $j(\mu) = \mu$ because $\mu < \kappa$.

$\langle R(\alpha+2), \varepsilon, F \mid \alpha \rangle \vDash \mu$ is the last ordinal $< \kappa$ in the image of F, if there is such an ordinal and 0 otherwise. Hence, $\langle R(\beta+2), \varepsilon, F \mid \beta \rangle \vDash \mu$ is the last ordinal $< j(\kappa)$ in the image of F if there is such an ordinal and 0 otherwise.

Let n_0 be the last such that $j^{n_0}(\kappa)$ is defined. Exactly as in the proof of Lemma 2, we can prove by induction for $i \leqq n_0$.

$\langle R(\alpha), \varepsilon, F \cap \alpha \rangle \vDash \mu$ is the last ordinal $< j^i(\kappa) \cap \alpha$ in the image of F if there is such an ordinal and 0 otherwise. But $j^{n_0}(\kappa) \geqq \alpha$. Therefore, $j^{n_0}(\kappa) \cap \alpha = \alpha$.

$\langle R(\alpha+2), \varepsilon, F \mid \alpha \rangle \vDash \mu$ is the last ordinal $< \alpha$ in the image of F.

Hence, the image of F is bounded in α, but F is properly increasing and $\alpha \in C$ implies that α is closed under F — a contradiction! Thus κ is a fixed point of F.

To prove that (V') does not imply (V), it is enough to show that $\langle R(\kappa), \varepsilon \rangle$ (κ as before, does not matter who F is) is a model of $ZFC + (V')$. Define a normal ultrafilter U on κ by

$$A \in U \leftrightarrow \kappa \in j(A).$$

The set $A = \{\delta \mid \delta < \kappa,\ \delta$ is γ-extendible for all $\gamma < \kappa\}$ is in U for $j(A) = \{\delta \mid \delta < j(\kappa), \delta$ is γ-extendible for all $\gamma < j(\kappa)\}$.

$\kappa \in j(A)$ because κ is extendible.

By Lemma 5 being δ extendible for $\delta < \kappa$ relativizes to $R(\kappa)$.

Because U is normal, any normal function on κ has got a fixed point in A which implies $\langle R(\kappa), \varepsilon \rangle$ is a model of $ZFC + (V')$ (κ is inaccessible). Q.E.D.

Actually, we can prove (by modifications of the arguments in Theorem 3) that (K) is equivalent to the following schema: where by "$R(\alpha)$ reflects V with respect to τ" we mean "for all $x \in R(\alpha)$ $R(\alpha) \vDash \tau(x) \leftrightarrow V \vDash \tau(x)$.

(S) There exists unbounded number of β's such that $\langle R(\beta), \varepsilon \rangle$ reflects V with respect to $\tau(x)$ and there is $\kappa < \beta$, supercompact and a n.u.f. over $P_\kappa(\beta)$ s.t.

$M_u \vDash \beta$ reflects V with respect to τ.

We now prove that the existence of extendible cardinal is equivalent to a certain compactness theorem for second (or finite) order language.

Definition. Logic is called κ compact iff for every set of formulae A in this logic, if every subset of A of cardinality $< \kappa$ has a model, then A has a model.

The L_κ^n logic is like the n-th order logic, except that we allow conjunction and disjunctions of less than κ formulae. The usual second order logic is of course L_ω^2.

THEOREM 4. *κ is extendible iff L_κ^2 is κ compact. κ is the first extendible iff it is the first α such that second order logic is α compact.*

PROOF.

LEMMA 6. *If κ is extendible then L_κ^n $(n < \omega)$ is κ compact.*

PROOF. Let A be a set of formulae in L_κ^n such that for any subset of A of cardinality $< \kappa$ there is a model.

Let β be a sufficiently large cardinal such that:

a) $\kappa < \beta$.

b) $A \in R(\beta)$.

c) If c is a set of formulae in L_κ^n, $c \in R(\beta)$ and c has a model, then c has a model in $R(\beta)$.

d) $\mathrm{cf}(\beta) = \aleph_1$.

e) $|R(\beta)| = \beta$.

By definition of extendibility there is α and elementary embedding j of $R(\beta)$ into $R(\alpha)$ such that $j(\kappa) > \kappa$ and κ is the first ordinal moved by j.

Let n be the last n such that $j^n(\kappa)$ is defined. There is such n as shown by the proof of Lemma 2 in the case $\mathrm{cf}(\alpha) > \omega$.

We prove by induction on $i \leqq n$ that any subset of A of cardinality $< j^i(\kappa)$ has a model. For $i = 0$ that is our assumption about A. Assume it is true for i and $i + 1 \leqq n$. Clearly, $j^i(\kappa) < \beta$, thus by definition of β, $R(\beta) \models$. Any subset of A of cardinality $< j^i(\kappa)$ has a model. Thus, $R(\alpha) \models$. Any subset of $j(A)$ of cardinality $< j^{i+1}(\kappa)$ has a model.

Let B be a subset of A of cardinality $< j^{i+1}(\kappa)$. Let $B' = \{j(\phi) \mid \phi \in B\}$. B' is clearly a subset of $j(A)$ of cardinality $< j^{i+1}(\kappa)$. $B' \in R(\alpha)$; thus B' has a model. Since α is a limit ordinal, the model of B' in $R(\alpha)$ is a model in the universe. Any formula of B is in L_κ^n and thus of length $< \kappa$. It follows that if B' has a model then B has a model too — because the formulae of B' are like those of B except perhaps of renaming the predicates appearing in them. Thus we proved our claim. $|A| < \beta \leqq j^n(\kappa)$. Thus, A has a model. ($|A| < \beta$ because $|R(\beta)| = \beta$).

Now we prove the other direction of the theorem.

Let $\kappa < \beta$. Take a constant c_x for each $x \in R(\beta)$, an additional constant c, a binary predicate symbol E (to be interpreted as ε) and a unary predicate symbol

K (to be interpreted as κ). Let Φ be the Π_1^1 statement asserting that E is a well founded relation. Let CD be the complete diagram of $\langle R(\beta), \varepsilon, \kappa\rangle$ in first order language (that is, all formulae in the language of E, K and $\{c_x \mid x \in R(\beta)\}$ true in $\langle R(\beta), \varepsilon, \kappa\rangle$ if we realize c_x by x).

Define:

$$A = CD \bigcup \{\chi, \Psi, \Phi\} \bigcup \{c > c_\alpha \mid \alpha < \kappa\} \bigcup \{K(c)\} \bigcup \{xEc_\alpha \rightarrow \bigvee_{\beta < \alpha} x = c_\beta \mid \alpha < \kappa\},$$

where χ and Ψ are the statements defined in the proof of Theorem 2.

Let A' be a subset of A of cardinality $< \kappa$. Without loss of generality, assume $\{\chi, \Psi, \Phi, K(c)\} \subseteq A'$. Let α be $\sup \{\beta \mid C_\beta$ appears in some formula in A', $\beta < \kappa\}$.

$\alpha < \kappa$ since $|A'| < \kappa$ and κ is inaccessible, since it is strongly compact (i.e., L_κ^1 is κ compact). We shall now see that $\mathscr{L} = \langle R(\beta), \varepsilon, \kappa, \{x\}_{x \in R(\beta)}, \alpha\rangle$ is a model of A'. (We take α as the interpretation of c and x as interpretation of c_x.)

$\mathscr{L}$ is obviously a model of CD. As noted in the proof of Theorem 2; $\mathscr{L}$ is a model of Ψ and χ. It is trivial that $\mathscr{L}$ satisfies $K(c)$ and Φ, and

$$\{xEc_\gamma \rightarrow \bigvee_{\beta < \gamma} x = c_\beta \mid \gamma < \kappa\}.$$

But by definition of α, $A' \cap \{c > c_\alpha \mid \alpha < \kappa\}$ holds. Thus, any subset of A of cardinality $< \kappa$ has a model. By the hypothesis of the Theorem A has a model. Since it is a model of χ, Ψ, Φ it is isomorphic to some $\langle R(\alpha), \varepsilon, \rho, \{d_x\}_{x \in R(\beta)}, d\rangle$. (See the note about χ, Ψ in the proof of Theorem 2.)

Clearly, $d_\kappa = \rho$, $d_\alpha = \alpha$ for $\alpha < \kappa$, $d \in \rho$. Thus $\rho > d > d_\alpha$; thus, $\rho > \kappa$.

The map $x \rightarrow d_x$ is clearly an elementary embedding of $\langle R(\beta), \varepsilon\rangle$ into $\langle R(\alpha), \varepsilon\rangle$ and κ is the first ordinal moved by this map.

THE PROOF OF THE SECOND PART OF THEOREM 4. Let κ be the first ordinal such that second order logic (L_ω^2) is κ compact. We shall prove that there is an extendible cardinal $\leqq \kappa$ which is all we have to prove (since the other direction follows trivially from the first part of the Theorem).

Let β be an ordinal such that:

(a) $\beta > \kappa$.

(b) If $\alpha < \beta$ and there is $\gamma > \alpha$ such that α is not γ extendible, then there is such $\gamma < \beta$.

Again pick a constant c_x for each $x \in R(\beta)$, an additional constant c and unary and binary predicate symbols E and K.

Define $A = CD \cup \{\chi, \Psi, \Phi\} \cup \{c > c_\alpha \mid \alpha < \kappa\} \cup \{K(c)\}$, where χ, Ψ, Φ, CD are as in the proof of the first part of the Theorem. The same proof gives that there is α and an elementary embedding j of $\langle R(\beta), \varepsilon \rangle$ into $\langle R(\alpha), \varepsilon \rangle$ such that $j(\kappa) > \kappa$. However, we do not know now that κ is the first ordinal moved by j.

Let δ be the first ordinal moved by j. δ is β extendible by definition, and thus it is γ extendible for each $\gamma < \beta$. Then, by definition of β, δ is extendible. This proves the existence of an extendible cardinal $\leqq \kappa$.

References

1. K. Kunen, *Elementary embeddings and infinitary combinatorics*, (to appear).

2. M. Magidor, *There are many normal ultrafilters corresponding to a supercompact cardinal*, Israel J. Math. **9** (1971), 186–192.

3. Motangue-Vaught, *Natural models of set theory*, Fund. Math. **47** (1959), 219–242.

4. W. N. Reinhardt and R. Solovay, *Strong axioms of infinity and elementary embeddings*, (to appear).

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